Description of Real AW*- Factors of Type I

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A B S T R A C T
The paper of the is to initiate the study of real AW-algebras in the framework of the theory of real C*-algebras and W*-algebras. It happens that in some aspects real AW*-algebras behave unlike complex AW*-algebras and sometimes their properties are completely di□erent also from corresponding properties of real W*-algebras. We prove that if the complexification $A+iA$ of a real C*-algebra A is a (complex) AW*-algebra then A itself is a real AW∗-algebra. By modifying the Takenouchi's examples of complex non-W*, AW*-factors we show that there exist real non-W*, AW*-factors. The correspondence between real AW*-factors and involutive (i.e. with period 2) *-anti-**automorphisms** of (complex) AW*-factors is established. We give the decomposition of real AW*-algebras into types I, II and III similar to the case of complex AW*-algebras or W*-algebras. It is proved that if A is a real AW*-factor and its complexification $M = A+iA$ is also an AW*-algebra (and therefore an AW*-factor) thenthetypesof A and M coincide.*

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1. INTRODUCTION

The theory of operator algebras was initiated in a series of papers by Murray and von Neumann in thirties. Later such algebras were called von Neumann algebras or W*-algebras. These algebras are self-adjoint unital subalgebras *M* of the algebra *B*(*H*) of bounded linear operators on a complex Hilbert space *H*, which is closed in the weak operator topology. Equivalently *M* is a von Neumann algebra in *B*(*H*) if it is equal to the commutant of its commutant (von Neumann's bicommutant theorem). A factor (or W*-factor) is a von Neumann algebra with trivial center and investigation of general W*-algebras can be reduced to the case of W*-factors, which are classified into types *I*, *II* and *III*.

Real operator algebra is a *-algebra consisting of bounded (real) linear operators on a real Hilbert space *H*. If it is closed in the weak operator topology we have real W*-algebra, and if it is uniformly closed (i.e. in the norm topology) then we come to the notion of the real C*-algebra.

In his monograph [7] Li Bing-Ren has set up the fundamentals of real operator algebras and gave a systematic discussion of the real counterpart for the theory of W*- and C*-algebras.

A slightly different (but almost the same up to *-isomorphism) definition of real W*-algebras was given by E.Størmer [13,14]: A real von Neumann algebra (or real W*-algebra) is a real *-algebra $\mathfrak{R}$ of bounded linear operators on a complex Hilbert space containing the identity operator **1**, which is closed in the weak operator topology and satisfies the condition $\mathfrak{R} \cap i\mathfrak{R} = \{0\}$. The smalles (complex) von Neumann algebra $U(\mathfrak{R})$ containing $\mathfrak{R}$ coincides with its complexification $\mathfrak{R} \cap i\mathfrak{R}$, i.e. $U(\mathfrak{R}) = \mathfrak{R} \cap i\mathfrak{R}$. Moreover $\mathfrak{R}$ generates a natural involutive (i.e. of order 2) *-antiautomorphism $\alpha_{\mathfrak{R}}$ of $U(\mathfrak{R})$, namely $\alpha_{\mathfrak{R}} = (x+iy) = x^* + iy^*$, where $x+iy \in U(\mathfrak{R}), x, y \in \mathfrak{R}$. It is clear that $\mathfrak{R} = \{x \in U(\mathfrak{R}) : \alpha(x) = x^*\}$.

Conversely, given a (complex) von Neumann algebra U and any involutive *-antiautomorphism α on U, the set $\{x \in U : \alpha(x) = x^*\}$ is a real von Neumann algebra in the above sense.

It is not diffucul to see that two real von Neumann algebras generating the same (complex) von Neumann algebra are isomorphic if and only if the corresponding involutive *-antiautomorphisms are conjugate. Thus the study of the above real von Neumann algebra can be reduced to the study of pairs (U, α), where U is a (complex) von Neumann algebra and α - its involutive *-antiautomorphism.

2. PRELIMINARIES

Let H be a complex Hilbert space, $B(H)$ denote the algebra of all bounded linear operators on H. The *weak (operator) topology* on $B(H)$ is the locally convex topology, generated by semi norms of the form: $\rho(a) = |(\xi, a\eta)|$, $\xi, \eta \in H$, $a \in B(H)$. W*-algebra is a weakly closed complex *-algebra of operators on a Hilbert space H containing the identity operator **1**. Recall that W∗-algebras are also called *von Neumann algebras*.

Let further M be a W*-algebra. The set M' of all elements from $B(H)$ commuting with each element from M is called the commutant of the algebra M. The *center* $Z(M)$ of a W*-algebra M is the set of elements of M, commuting with each element from M. It is easy to see that $Z(M) = M \cap M'$. Elements of $Z(M)$ are called *central* elements. A W*-algebra M is called *factor*, if $Z(M)$ consists of the complex multiples of **1**, i.e if $Z(M) = \{\lambda 1, \lambda \in \mathbb{C}\}$. We say that a W*-algebra M is *injective* if thereexists a projection P in $B(H)$ onto M such that $\|P\| = 1$ and $P(1) - \mathbf{1}$. This isequivalent to the condition that M is *hyperfinite*, i.e., that there exists an increasing sequence $\{M_n\}$ of matrix subalgebras of the algebra M containing **1** and such that the union $\cup_n M_n$ is weakly dense in M.

Let e, f, h be projections from M. We say that e is equivalent to f, and write $e \sim f$, if $e = \omega^* \omega$, $f = \omega\omega^*$ for some partial isometry ω from M. A projection e is called: *finite*, if $e \sim f \le e$ implies $f = e$; *infinite* - otherwise; *purely infinite*, if e doesn't have any nonzero finite subprojection; abelian, if the algebra eMe is an abelian W∗-algebra. A W∗-algebra M is called *finite, infinite, purely infinite*, if **1** is a finite, infinite, purely infinite respectively; M is σ-finite, if any family of pairwise orthogonal projections from M is at most countable; *semifinite*, if each projection in M contains a nonzero finite subprojection; *properly infinite*, if every nonzero projection from $Z(M)$ is infinite; *discrete*, or of type *I*, if it contains a faithful abelian projection (i.e. an abelian projection with the central support **1**); *continuous*, if there is no abelian projection in M except zero; M is of type *II*, if M is semifinite and continuous; type I_{fin} (respectively I_∞), if M is of type *I* and finite (respectively properly infinite); type II_1 (respectively type II_∞), if M is of type *II* and finite (respectively properly infinite); type *III*, if M is purely infinite. It is known that (see for example [103]) any W*-algebra has a unique decomposition along its center into the direct sum of W*-algebras of the I_{fin}, I_∞, II_1, II_∞ and *III* types.

A linear mapping $\alpha : M \to M$ is called a **-automorphism* (respectively a *-antiautomorphism) if $\alpha(x^*) = \alpha(x)^*$ and $\alpha(xy) = \alpha(x)\alpha(y)$ (respectively $\alpha(xy) = \alpha(y)\alpha(x)$, for all $x, y \in M$. A mapping α is called involutive if $\alpha^2 = id$. A *-automorphism α is called *inner* if there exists a unitary u in M, such that $Adu(x) = uxu^*$, for all $x \in M$. A *-automorphism is called centrally trivial if $\alpha(x_n) - x_n \to 0$ *-strongly as $n \to \infty$ for any central sequence $\{x_n\}_{n \in \mathbb{N}}$. We shall denote by $Aut(M)$ the group of all *-automorphisms, by $Aut(M)$ the group of all *antiautomorphisms, by $Int(M)$ the group of all inner *-automorphisms, and by Ct(R) the subgroup of its centrally trivial *-automorphisms of M. Two *-

automorphisms or *-antiautomorphisms α and β are said to be conjugate (or outer conjugate), if $\alpha = \theta \cdot \beta \cdot \theta^{-1}$ (respectively $Adu \cdot \alpha = \theta \cdot \beta \cdot \theta^{-1}$) for some *-automorphism θ (and an inner *-automorphism *Adu*). A linear functional ω on *M* is called *positive*, if $\omega(x^* x) \geq 0$ for all $x \in M$. A positive linear functional ω on *M* with $\|\omega\| = 1$ is called a state. Let M_+ be the positive part of *M*. A *weight* on *M* is a homogeneous additive function $\omega : M_+ \to [0, +\infty]$ (we suppose that $0 \cdot +\infty - 0$). A weight (or a state) ω is called: *faithful*, if for any $x \in M_+, \omega(x) = 0$ implies $x = 0$; *normal*, if for any net $\{x_\alpha\}$ in *M*, increasing to an element *x*, we have $\omega(x) = \sup_\alpha \omega(x_\alpha)$; *finite*, if $\omega(x) < \infty$ for all $x \in M_+$; *semifinite*, if for any $x \in M_+$ there exists a net of elements $\{y_\alpha\} \in M_+$, such that $\omega(y_\alpha) < \infty$, and $y_\alpha \to x$ in σ - weak topology; ω is a trace, if $\omega(uxu^*) = \omega(x)$ for all $x \in M_+$ and each unitary $u \in M$.

The type of a W*-algebra is tightly connect with the existence of traces on it. Namely a W*-algebra *M* is a finite if and only if it possesses a separating family of finite normal traces; it is semifinite if and only if it possesses a faithful semifinite normal trace; *M* is purely infinite if and only if there is no nonzero semifinite normal trace on M (see [15]).

Definition. [4]. By a real C*-algebra we mean a real Banach *-algebra *R* such that the relation $\|a^* a\| = \|a\|^2$ holds and the element $1 + a^* a$ is invertible for any $a \in R$.

Definition*. [8,9]. A real C*-algebra *R* such that *R*+*iR* is a complex W*-algebra is referred to as a real W*-algebra.

We proceed with another definition of a real W*-algebra, which can be found in papers of Størmer.

Definition. [2,13].** A unital weakly closed real *-algebra *R* in $B(H)$ such that $R \cap iR = \{0\}$ is called a *real W*-algebra*.

A real W*-algebra *R* is called a (real) factor if its center *Z*(*R*) consists of elements $\lambda \mathbf{1}$, $\lambda \in \mathbb{R}$. We say that a real W*-algebra *R* is of type $I_{fin}, I_\infty, II_1, II_\infty$ and *III*, $\lambda \in [0,1]$ if the enveloping W*-algebra $U(R) = R + iR$ (i.e., the least W*-algebra containing *R*) is of the corresponding type with respect to the usual classification of W*-algebras.

3. ON CONSTRUCTION OF REAL AW*-FACTORS

Let A be a real C*-algebra, with the complexification $M = A + iA$. Then *M* is a complex C*-algebra and, as we have seen in the previous section, if A is a real AW*-algebra *M* may not be a (complex) AW*-algebra. Now let us consider the converse problem if $M = A + iA$ is an AW*-algebra is A necessarily a real AW*-algebra? The following result gives a positive answer to this problem.

Proposition 4.3.1 *Let A be a real C*-algebra and let $M = A + iA$ be its complexification. Suppose that M is an AW*-algebra. Then A is a real AW*-algebra.*

Proof. As we have mentioned in the first section, A coincides with the fixed point set under the conjugate linear *-automorphism $"-" : x - iy \mapsto x - iy$ of *M*, where $x, y \in A$, i.e. $A = \{a \in M : \overline{a} = a\}$. If *S* is a nonempty subset in A then for its right-annihilator (with respect to *M*) we have

$$R_M(S) = \{a \in M \,|\, sa = 0 \text{ for all } s \in S\}$$

and

$$a \in R_M(S) \Leftrightarrow sa - 0, \forall s \in S \Leftrightarrow \overline{sa} - \overline{s}\,\overline{a} - s\overline{a} - 0, \forall s \in S$$

because $\overline{s} = s \in A$. This means that $a \in R_M(S)$ if and only if $\overline{a} \in R_M(S)$.

Now suppose that M is an AW*-algebra, then $R_M(S) = gM$ for a suitable projection $g \in M$. Since $g \in R_M(S)$ from above it follows that $\overline{g} \in R_M(S)$. Therefore $\overline{g}$ is a projection and $\overline{g} \in gM$, i.e. $\overline{g} = g\overline{g}$. Thus $(\overline{g})^* = (g\overline{g})^* = (\overline{g})^* g^* = \overline{g}g = \overline{g} = g\overline{g}$ i.e. $g = \overline{=}{g} = \overline{g\overline{g}} = \overline{g}g = \overline{g}$

This means that $g \in A$. But then

$$R_M(S) = R_M(S) \cap A = gM \cap A = gA,$$

i.e. A is a real AW*-algebra.

Proposition 4.3.4 *There exist real AW*-factors which are not real W*-factors.*

Theorem 4.3.5 *A real AW∗-algebra A is a real W*-algebra if and only if*

A. *A possesses a separating family of normal states;*

B. *its complexification* $M = A + iA$ *is an AW*-algebra.*

Proof. Necessity is obvious, since if A is a real W*-algebra, then $M = A + iA$ is a (complex) W*-algebra (see [7, Chap.5]). Therefore, M is an AW*-algebra and it possesses a separating family of normal states, the restrictions of which on A give a separating family of normal states on A.

Sufficiency. Let $M = A + iA$ be an AW*-algebra and let A possess a separating family of normal states, which we denote by $\{f_\gamma\}$, i.e. for any $a \in A, a \geq 0, a \neq 0$ exists $f \in \{f_\gamma\}$ with $f(a) = 0.$

For $x \in a + ib \in M, a, b \in A,$ we put $\alpha(x) = a^* + ib^*.$ A straightforward calculation shows that α is an involutive (i.e. with period 2) *-anti-automorphism of M, and $A = \{a \in M : \alpha(a) = a^*\}$.

The extension of f_γ by linearity on M we denote by f_γ^0, and we shall show, that the family $\{f_\gamma^0\}$ is a separating family of normal states on M.

For $x - a + ib \in M_s - \{x \in M : x^* - x\}$ we have $a^* = a, b^* = -b$, and since f_γ is hermicitian we obtain $f_\gamma^0(x) = f_\gamma(a) + if_\gamma(b) = f_\gamma(a),$ since $f_\gamma(b) = 0$. Thus, for $x \in M_s$ we have

$$f_\gamma^0(x) = \frac{1}{2} f_\gamma(x + \alpha(x)),$$

and $x + \alpha(x) \in A,$ since $x + \alpha(x) = 2a.$

If $x \geq 0$, then $\alpha(x) \geq 0$ (because α is a *-anti-automorphism), and hence $x + \alpha(x) \geq 0,$ i.e. $x + \alpha(x) \in A^+$. Therefore $f_\gamma^0(x) = \frac{1}{2} f_\gamma(x + \alpha(x)) \geq 0$, i.e. all functionals f_γ^0 are positive on M. Moreover, we have $f_\gamma^0(1) = f_\gamma(1) = 1$, i.e. $\{f_\gamma^0\}$ is a family of states on M.

Now let us show, that each state f_γ^0 is normal. If $\{x_\nu\} \subset M$ is an arbitrary net with $x_\nu \searrow 0,$ then since α is an order isomorphism of M, we have $\alpha(x_\nu) \searrow 0.$ Therefore $x_\nu + \alpha(x_\nu) \searrow 0$ and $x_\nu + \alpha(x_\nu) \in A^+$. Since f_γ is a normal we obtain

$$f_\gamma^0(x_\nu) = \frac{1}{2} f_\gamma(x_\nu + \alpha(x_\nu)) \to 0$$

i.e. all functionals f_γ^0 are normal on M.

Finally, let $x \in M, x \geq 0$ and $f_\gamma^0(x) = 0$ for all γ. Then $x + \alpha(x) \in A^+$, and since $\{f_\gamma\}$ is a separating family of states, $x + \alpha(x) = 0$. Hence we have $x = -\alpha(x) \in M^- \cap (-M^+) = \{0\}$, i.e. $x = 0$. Thus, the AW*-algebra *M* possesses a separating family of normal states $\{f_\gamma^0\}$. By the theorem of Pedersen [10] *M* is a W∗-algebra. Therefore, by [7] *A* is a real W∗-algebra.

Now, let *M* be a (complex) AW*-factor, *α* its involutive *-anti-automorphism. Then as it was mentioned above the set $A = \{\alpha \in M : \alpha(a) = a^*\}$ is a real C*-algebra such that $M = A + iA$ (actually $\overline{x} = \alpha(x^*)$ in terms of operation "-") and from Proposition 4.3.1 it follows that *A* is a real AW*-factor. It is known that two real W*-algebras generating the same (complex) W*-algebra, are isomorphic if and only if the corresponding involutive *-anti-automorphisms are conjugate [2,13,14]. A similar result is also valid for real AW*-algebras:

Proposition 4.3.8 *Let α and β be involutive *-anti-automorphisms of a (complex) AW*-factor M. Then the real AW*-factors* $A = \{x \in M : \alpha(x) = x^*\}$ *and* $B = \{x \in M : \beta(x) = x^*\}$ *are real *-isomorphic if and only if the involutive *-anti-automorphisms α and β are conjugate, i.e.* $\beta = \theta\alpha\theta^{-1}$ *for a suitable *-automorphism of the AW*-factor M.*

Proof. Let *A* and *B* be real *-isomorphic with a *-isomorphism $\theta_0 : A \mapsto B$. Then θ_0 can be naturally extended to a (complex) *-isomorphism *θ* of their complexifications $A + iA$ and $B + iB$ both coincide with *M*. Therefore *θ* is a *-automorphism of *M* and $\theta(A) = B$, i.e. $\alpha(x) = x^*$ if and only if $\beta(\theta(x)) = (\theta(x))^* = \theta x^*$. Thus for $x \in A$ we have $\beta(\theta(x)) = (\theta(x))^* = \theta(x^*) = \theta\alpha(x)$, i.e. $\beta = \theta\alpha\theta^{-1}\alpha(x)$ for all $x \in A$.

Since $\theta^{-1}\beta^{-1}\theta\alpha$ is a *-automorphism on *M* which is identical on *A* and any real *automorphism of *A* can be uniquely extended to a complex *-automorphism of *M*, it follows that $\theta^{-1}\beta^{-1}\theta\alpha = id$ on whole *M*, i.e. $\theta\alpha = \beta\theta$ and $\beta - 0\alpha 0^{-1}$, i.e. *α* and *β* are conjugate.

Conversely, if *α* and *β* are conjugate, i.e. $\beta = \theta\alpha\theta^{-1}$ for a suitable complex *-automorphism *θ* of *M*, then $\theta\alpha = \beta\theta$ and $\alpha(x) = x^*$ if and only if $\beta(\theta(x)) = \theta x^* = (\theta x)^*$, i.e. $\theta(A) = B$. Therefore, *θ* restricted on *A* gives the needed *-isomorphism between real AW∗-factors *A* and *B*.

4. REAL AW*-ALGEBRAS OF TYPE *I*

Now we consider one of the main results of this section.

Theorem 4.5.2 *Let A be a real AW*-algebra and its complexification* $M = A + iA$ *is a (complex) AW*-algebra. Then A is of type I if and only if M of type I.*

Corollary 4.5.3 *Let A be a real AW*-algebra of type I, and its complexification* $M = A - iA$ *is a (complex) AW*-algebra. Then A is a real W*-algebra if and only if its center* Z_A *is a real W*-algebra.*

Proof. If *A* is a real W*-algebra then, obviously, its center Z_A is a real W*-algebra.

Conversely, let *A* be an AW*-algebra of type *I* and its center is a W*-algebra. Then by Theorem 4.5.2, $M = A + iA$ is an AW*-algebra of type *I*, and its center $Z_M = Z_A + iZ_A$ is a W*-algebra. From Kaplanskys theorem [6, Theorem 2] it follows that *M* is an W*-algebra. Therefore, *A* is a real W*-algebra.

5. MAIR RESULTS

Proposition 4.6.1 *Let A be a real C*-algebra, and let* $A_s = \{x \in A : x^* = x\}$ *and* $A_k = \{x \in A : x^* = x\}$. *If an element* $x \in A_s$ *commutes with all elements of* A_s, *then* $x \in Z_A$.

Proof. Let $x \in A_s$, and let $[x,a] = 0$ for any a ∈ As. If $b \in A_k$, then it is clear that $[x,b]^* = [x,b] \in A_s$, and, therefore, $[x,[x,b]] = 0$. In this case, it follows from the Putnam Theorem [11] (see also [12, Corollary 2.2.10]) that $[x,b] = 0$. Since $A = A_s \oplus A_k$, this means that $[x,y] = 0$ for any $y \in A$, i.e., $x \in Z_A$.

Hence we obtain the following corollary

Corollary 4.6.2 *In any real C*-algebra A, one has the relation* $[Z_A]_s = Z_{A_s}$.

Proposition 4.6.4 Let *e* be a semi-abelian projection in a real AW*-algebra *A*, and let *f* be a projection in *A*, where $f \le e$. Then $f = he$ for some central projection $f \in A$.

Proof. One has $f \in (eAe)_s$, and $f \in Z_{(eAe)_s}$, because $(eAe)_s$ is commutative. Hence $f \in Z_{(eAe)_s}$ by Proposition 4.6.1. Then it follows from [3, Sec. 6, Proposition 4] that $f = he$ for some central projection $h \in Z_A$.

Hence we obtain the following corollary

Corollary 4.6.5 *Let A be a real AW*-algebra containing only two central projections,* 0 *and* **1** *(in particular, A is a real AW*-factor). If e is a semi-abelian projection in A, then e is a minimal projection (an atom) in A.*

Proof. If $f \le e$, then $f = he$ by Proposition 4.6.4, where h is a projection in Z_A. By assumption, either $h=0$ or $h=\mathbf{1}$, i.e., either $f=0$ or $f=e$.

Theorem 4.6.6 *Let A be a real AW*-algebra having only two projections,* 0 *and* **1**. *Then A is *-isomorphic to* $\mathbb{R}$, *to* $\mathbb{C}$ *or to quaternions algebra* **H**.

Proof. It follows from the axioms of Baer *-ring that the right annihilator $R(x) = \{y \in A : xy = 0\}$ is equal to either 0 or *A* for any element x $x \neq 0$ in *A*. However, $\mathbf{1} \notin R(x)$, and, therefore, $R(x) = \{0\}$ for any $x \neq 0$. Thus, A has no zero divisors $(xy \neq 0, \ \forall x \neq 0, y \neq 0)$. By [7, Proposition 5.2.2], every element $a \in A_s$ of a real C*-algebra *A* has the form $a = a_+ - a_-$, where $a_+, a_- \in A^+, a_+ a_- = 0$.

Hence, either $a_+ = 0 \ or \ a_- = 0$, i.e., for any $a \in A_s$, the element a is either positive or negative (i.e., $-a \in A^+$).

Let us show that every element $a \in A^+$ is invertible. If $0 < a < \frac{1}{n}\mathbf{1}$, $\forall n = 1,2,...,$ then it is clear that $\|a\| \le \frac{1}{n}, n = 1,2,...,$ i.e., *a=0*. Therefore, if $a \in A^-, a \neq 0,$ then there is a positive integer n such that $\left(\frac{1}{n}\right)\mathbf{1} - a \in A^+$. In this case, $-\left(\frac{1}{n}1 - a\right) = a - \frac{1}{n}1 \in A^+$, i.e., $na - \mathbf{1} \in A^+$. This means that $na\mathbf{1} = x^* x$ for some $x \in A$. By an axiom of real C*-algebra, the element $na = \mathbf{1} + (na + \mathbf{1}) = \mathbf{1} + x^* x$ is always invertible, i.e., the element a is invertible.

If *b* is an arbitrary nonzero element of *A*, then $b^*b \geq 0$ and $bb^* \geq 0$ are nonzero positive elements, and, therefore, both these elements are invertible, i.e., there are elements c_1 and c_2 such that $c_1 b^* b = \mathbf{1}$ and $bb^* c_2 = \mathbf{1}$, i.e., b has a right inverse ($b^* c_2$) and a left inverse ($c_1 b^*$), i.e., *b* is invertible.

Thus, *A* is a real C*-algebra which is a division algebra. By [7, Theorem 2.2.2], the algebra *A* is isomorphic to $\mathbb{R}$, to $\mathbb{C}$ *or to quaternions algebra* **H**. Moreover, it follows from the uniqueness of the *-operation in a real C*-algebra [7, Theorem 5.6.5] that this isomorphism is a *-isomorphism.

Corollary 4.6.7 *Let A be a real AW*-algebra containing only two central projections,* 0 *and* **1**. *Then either* $Z_A = \mathbb{R}\mathbf{1}$ *or* $Z_A = \mathbb{C}\mathbf{1}$.

Proof. The proof follows from Theorem 4.6.6 applied to the commutative AW*-algebra Z_A.

Remark 4.6.8 Corollary 4.6.7 shows that if a real AW*-algebra *A* contains only two central projections, 0 and **1**, i.e., if *A* is a Baer *-factor in the sense of [3], then *A* is either a real AW*-factor (i.e., $Z_A = \mathbb{R}\mathbf{1}$) or a complex AW*-factor, because $Z_A = \mathbb{C}\mathbf{1}$ and *A* is a *-algebra over $\mathbb{C}$. For this reason, in what follows, we refer to a real AW*-algebra *A* such that Z_A contains only two projections, 0 and **1**, as an AW*-factor, and this notion includes real AW*-factors and complex AW*-factors as special cases.

Corollary 4.6.9 *A projection e in an AW*-factor A is semi-abelian if and only if eAe coincides with* $\mathbb{R}\mathbf{1}$, $\mathbb{C}\mathbf{1}$ *or to quaternions algebra* **H1**.

Proof. If *e* is semi-abelian, then *e* is an atom, and hence the algebra *eAe* contains only two projections, 0 and *e*. By Theorem 4.6.6, the algebra *eAe* coincides with $\mathbb{R}\mathbf{1}$, $\mathbb{C}\mathbf{1}$ or to quaternions algebra **H1**. The converse assertion follows from the fact that the algebra $(\mathbb{R}e)_s = (\mathbb{C}e)_s = (\mathbf{H}e)_s = \mathbb{R}\mathbf{1}$ is commutative.

Corollary 4.6.10 *A projection e in an AW*-factor A is semi-abelian if and only if it is minimal.*

Theorem 4.6.11 *Let A be an AW*-factor of type I. Then A is isometrically *-isomorphic to the algebra B(H) of all bounded* **P***-linear operators on the Hilbert space H over* **P**, *where* **P** *is equal to* $\mathbb{R}$, $\mathbb{C}$ *or* **H**. *In particular, every AW*-factor of type I is a W*-factor (real or complex).*

Proof. Let $e \in A$ be a nonzero semi-abelian projection in *A*. Obviously, *eA* is a right ideal in *A*. Let us show (following [5, Theorem 7.3]) that the right ideal *eA* can be equipped with the structure of a Hilbert space over $\mathbb{R}$, $\mathbb{C}$ or **H**. For $\xi, \eta \in eA = H$, say, $\xi = ea$ and $\eta = eb$, consider the element $\xi\eta^* = eab^*e$. Since $eAe = \{\lambda e, \lambda \in \mathbb{R}\}$, where **P** is equal to $\mathbb{R}$, $\mathbb{C}$ or **H** (Corollary 4.6.9), it follows that $\xi\eta^* = \lambda(\xi,\eta)e$, where $\lambda(\xi,\eta) \in \mathbf{P}$.

Write $\langle \xi, \eta \rangle := \lambda(\xi,\eta)$. We obtain a bilinear form on $H \times H$ with values in **P**. Moreover, $\langle \xi, \xi \rangle e = \xi\xi^* \geq 0, \ \forall \xi \in H$, i.e., $\langle \xi, \xi \rangle \geq 0$. If $\langle \xi, \xi \rangle = 0$, then $\xi\xi^* = eaa^*e = 0$, and it follows from an axiom of real C*-algebra (namely, from the formula $\|\xi\xi^*\| = \|\xi\|^2$) that $\xi=0$. Thus, $H=eA$ is a Hilbert space over **P** (which is equal to $\mathbb{R}$, $\mathbb{C}$ or **H**) with respect to the inner product introduced above. Moreover, the norm of an element ξ in the Hilbert space *H* coincides with the norm of ξ in the C*-algebra, because

$$\|\xi\|_H^2 e = \langle \xi, \xi \rangle e = \xi\xi^* = eaa^*e = \|(ea)(ea)^*\|e = \|ea\|^2 e = \|\xi\|^2 e$$

i.e. $\|\xi\|_H = \|\xi\|$. Now, let us define an action of the AW*-factor *A* on $H=eA$ as follows: consider the representation $x \to R_x : H \to H, \ x \in A$ given by $R_x(\xi) = R_x(ea) := eax = \xi x$. One can readily see that R_x

is a **P** -linear operator on *H*, namely, $R_x(\lambda\xi) = (\lambda\xi)x = \lambda(\xi x) = \lambda R_x(\xi), \quad \lambda \in \mathbf{P}$ $R_x(\xi+\eta) = (\xi+\eta)x = \xi x + \eta x = R_x(\xi) + R_x(\eta)$ Further, $R_{x+y} = R_x + R_y, \quad R_{xy} = R_x R_y, \quad R_{\lambda y} = \lambda R_y,$ and hence the mapping $x \to R_x$ is a representation of *A* by bounded operators on *H*. Moreover, this representation is faithful. Indeed, if $R_x = 0, x \in A,$ then $eax = 0$ for any $a \in A$, i.e., $eAx = \{0\}.$ Since *A* is a Baer *-factor, it follows from [26, Sec. 6, Corollary 2, Proposition 3] that either *e*=0 (which is impossible) or *x*=0. Hence $R_x = 0 \Rightarrow x = 0$. Thus, *A* admits a faithful representation by bounded **P**-linear operators on the Hilbert space *H*, i.e., A can be embedded *-isomorphically in *B*(*H*) (the real W*-algebra of all bounded **P**-linear operators on the Hilbert space *H*). Since a C*-norm is unique [7, Sec. 5], this embedding is isometric.

Let us now show that here all one-dimensional projections in *B*(*H*) belong to *A*. Indeed, if $\xi - ea \in H, \ \xi \neq 0,$ then, as is known, the one-dimensional projection onto the subspace generated by ξ is of the form

$$P_\xi(\eta) = \langle \eta, \xi \rangle \xi = \langle \eta, \xi \rangle e\xi = \eta \xi^* \xi = R_{\xi^*\xi}\eta,$$

i.e., the one-dimensional projections in *B*(*H*) are operators of the form $R_{\xi^*\xi}, \xi^*\xi \in A,$ and thus $P_\xi \in A$. Let us now show, following the idea of [6, Lemma 1], that every projection in *B*(*H*) belongs to *A*. If *p* is an arbitrary projection in *B*(*H*), then one can represent *p* as suppi, where pi are one-dimensional projections in *B*(*H*) belonging to *A*. Since the lattice of projections in a real AW*-algebra is complete (this remains valid in every Baer *-algebra [3, Sec. 4, Proposition 1]), it follows that the projection $q = \sup p_i$ exists in *A*. Since $A \subset B(H)$, it clearly follows that $q \geq p.$ If $q - p \neq 0,$ then $q - p$ is a nonzero projection in *B*(*H*), and hence there is a one-dimensional projection $q' \leq q - p$ which belongs to *A*, less than *q*, and is orthogonal to any p_i, and, therefore, is orthogonal to $q = \sup p_i$ as well; a contradiction. Hence $q = p \in A.$ Thus, *A* contains all projections in *B*(*H*). Since every hermitian operator in *B*(*H*) (as well as in any real W*-algebra) is a C*-norm limit of real linear combinations of projections and *A* is closed with respect to the C*-norm, it follows that A contains every hermitian operator in *B*(*H*), i.e., $A_s = B(H)$. If the AW*-factor *A* is complex, i.e., if **P**=$\mathbb{C}$, then *H* is a complex Hilbert space, and it is clear that

$$B(H) = B(H)_s + iB(H)_s = A_s + iA_s = A.$$

If *A* is a real AW*-factor, then **P**= $\mathbb{R}$ or **P**=**H**, and, therefore, *B*(*H*) is a real W*-factor. By [1, Theorem 3], every skew-Hermitean element in *B*(*H*) (except for the case in which *B*(*H*)=**H**, where one can apply Theorem 4.6.6) is a finite sum of commutators of skew-Hermitean elements, i.e.,

$$B(H)_k \subset [B(H)_s, B(H)_s] = [A_s, A_s] \subset A$$

Hence *A*=*B*(*H*).

Corollary 4.6.12 Every real AW*-factor of type *I* is a real W*-factor which is isomorphic to *B*(*H*), where *H* is a real or quaternionic Hilbert space.

Corollary 4.6.13 For any real AW*-factor of type *I*, the complexification $A_c = A + iA$ is always an AW*-factor.

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