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On OIM-Homeomorphisms and Quasiconformality

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A B S T R A C T
The main goal of studying these problems is to estimate the quasiconformality coefficient of a continued homeomorphism. It was proved that the OIM-homeomorphism is quasiconformal without requiring smoothness from the domain of definition to the boundary, and this quasiconformal reflection coefficient was estimated from the given ones.

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ENTRANCE

It is known that in space, a quasiconformal mapping f: $\mathbb{R}^n \to \mathbb{R}^n$ is fully characterized by the relationship between the moduli of families of curves Γ and the moduli of $\Gamma^* = f(\Gamma)$; for any family of curves Γ, the existence of a constant K such that the following inequalities hold:

$$\frac{1}{K} \cdot M(\Gamma) \le M(\Gamma^*) \le K \cdot M(\Gamma) \qquad (1)$$

defines a quasiconformal mapping. In particular, if we consider the values of a quasiconformal mapping defined on some continuum $\sum \subset R^n$, then for any continuum $E, F \subset \sum$, inequalities (1) hold for all families of curves Γ connecting the continua E and F. The "image" of such a family under the mapping f is clearly expressed. Here, the image of the family of curves Γ under the mapping f should be considered as the family of curves in space connecting the continua $f(E), f(F)$.

Definition 1 [1]. Let a homeomorphism $f : \sum \to^{\sim} {}^n$ be given ($\sum \subset R^n$). If for any disjoint continua $E, F \subset \sum$ and their images $f(\mathrm{E})$, $f(\mathrm{F})$, the following inequality holds:

$$\frac{1}{Q} \cdot M(\mathrm{E},\mathrm{F}) \le M(\mathrm{f}(\mathrm{E}),\mathrm{f}(\mathrm{F})) \le Q \cdot \mathrm{M}(\mathrm{E},\mathrm{F}),$$

then f is called an OIM-homeomorphism. The smallest Q is called the coefficient of the OIM-homeomorphism.

Definition 2 [2]. Consider the pair $<\sum,U>$, where E is a continuum, U is its neighborhood. If for any disjoint compact continua $E,F\subset\sum$ and their images $E^*,F^*\subset\sum^*$, the inequalities

$$\frac{1}{Q}M\left(E,F;U\right)\le M\left(E^*,F^*;U^*\right)\le QM\left(E,F;U\right)$$

hold, then the homeomorphism f is called an OIM-homeomorphism from the pair $<\sum,U>$ to the pair $<\sum^*,U^*>$.

Let the continua $\sum$ and $\sum^*$ lie compactly in the domains U and U^*, respectively. If the pairs $<\sum,U>$ and $<\sum^*,U^*>$ are quasiconformally equivalent, then the pairs $<\sum,\overline{R}^n>$ and $<\sum^*,\overline{R}^n>$ are quasiconformally equivalent [2].

Let a spherical metric be introduced in space $\tilde{}^n=\sim^n\cup\{\infty\},n\ge 2$. For a pair of continua $E_1,E_2\subset\tilde{}^n$, we denote by mod $M_n\left(E_1,E_2\right)$ the modulus of all families of curves in $\tilde{}^n$ connecting E_1,E_2, where E_1,E_2 are connected.

Definition 3 [1]. Suppose $\sum\subset\overline{R}^k$, $1<k\le n$ and $\omega:[0;+\infty)\to[0;+\infty)$ is a homeomorphism. A topological embedding $f:\sum\to\sim^n$ is called a $\omega_{n,k}$-embedding if for any disjoint pairs of continua $E_1,E_2\subset\sum$, the inequality

$$\frac{1}{\omega}\left(M_n\left(\mathrm{E}_1,\mathrm{E}_2\right)\right)\le \mathrm{M}_k\left(f\left(E_1\right),f\left(E_2\right)\right)\le\omega\left(M_n\left(E_1,\mathrm{E}_2\right)\right)$$

holds. If k = n, then the topological $\omega(t)=Kt$ embedding coincides with an OIM-homeomorphism.

LITERATURE ANALYSIS

[1], [2] the literature presents some results on the concepts of OIM-homeomorphism, topological localization, and OIM-homeomorphisms. [3] the literature presents a proof that OIM-homeomorphisms on surfaces are quasiconformal. [4] the literature presents a necessary and sufficient condition for a homeomorphism to be quasiconformal.

METHODOLOGY

$\sum,\sum^*\subset R^n$ - continua.

Theorem 1 [3]. Suppose $\sum,\sum^*\subset R^n$ are $(n-1)$-dimensional surfaces, $f:\sum\to\sum^*$ is an OIM-homeomorphism, Q is the coefficient of the OIM-homeomorphism, $a(\mathrm{n})>0$ is a constant depending on the dimension of the space. Then f is $K-$quasiconformal, where

$$K\le e^{\left[\frac{Q\omega_{n-1}}{a(\mathrm{n})}\right]^{\frac{1}{n-1}}}.$$

Lemma 1 [1]. If the mapping $f:\sum\to\sum^*$ is an OIM-homeomorphism, then the tangent mapping T_pf is also an OIM-homeomorphism.

Assume that $\sum,\sum'\in C^1$, i.e., M is a smooth surface.

Theorem 2 [1]. If the homeomorphism $f:\sum\to\sum^*$ is an OIM-homeomorphism, then it is $K-$quasiconformal with $K\le Q$.

Let a homeomorphism $f: D \to D'$ of domains be given. Then for f to be quasiconformal, it is necessary and sufficient that

$$\left\{ \limsup_{r \to 0} \frac{\max_{|x-x_0|=r} |f(x) - f(x_0)|}{\min_{|x-x_0|=r} |f(x) - f(x_0)|}, x \in D \setminus \{\infty, f^{-1}(\infty)\} \right\}$$

is bounded [4].

DISCUSSION AND RESULTS

Above, when studying the properties of OIM-homeomorphisms, we saw that the theorem holds when $\sum, \sum' \in C^1$. Now we attempt to resolve the question of whether the theorem holds when the surfaces are not smooth.

Suppose $\sum, \sum^* \subset R^n$ are given $(n-1)$-dimensional surfaces.

Theorem 3. If the homeomorphism $f: \sum \to \sum^*$ is a Q-OIM homeomorphism in space $\sim^n$, then f is $K-$ quasiconformal with $K \le e^{a(n)Q}$. Here, $a = a(n) > 0$.

Proof. For simplicity, we reduce the proof of the theorem to the case n = 3.

Let

$$B^3(x_0, r) = \{x \in R^3 : |x - x_0| \le r\},$$

$$B^3(r) = B^3(0, \mathrm{r}),\ B^3 = B^3(0,1),$$

$$S^2(x_0, r) = \{x \in R^3 : |x - x_0| = r\},$$

$$S^2(r) = S^2(0, r),\ S^2 = S^2(0,1).$$

Assume that $0 \in \sum, f(0) \in \sum^*$. For $0 < r < 1$, consider the sphere S^2.

Let

$$M = \max_{|x-x_0|=r} |f(x) - f(x_0)|,$$

$$m = \min_{|x-x_0|=r} |f(x) - f(x_0)|.$$

Now consider the quantity

$$L(r) = \frac{\max_{|x-x_0|=r} |f(x) - f(x_0)|}{\min_{|x-x_0|=r} |f(x) - f(x_0)|}$$

and estimate it from above.

Denote the preimages of the spheres $S^2(f(0), \mathrm{M})$ and $S^2(f(0), m)$ by S_1 and S_2, respectively. In $\sum$, we have $E = extS_1,\ F = \operatorname{int} S_2$. If γ_1 is a continuum connecting the point 0 with the sphere $S^2(0, r)$, and γ_2 is a continuum connecting the sphere $S^2(0, r)$ with the point at infinity, then the following inequality is obtained:

$$M\left(E,F;R^3\right)\geq M\left(\gamma_1,\gamma_2;\mathrm{R}^3\right)\geq a(3). \qquad (2)$$

By the property of the modulus of families of curves and the formulas for the moduli of annular domains, we obtain:

$$M\left(E^*,F^*;R^3\right)\leq M\left(B\left(f(0),\mathrm{m}\right),CB\left(f(0),M\right);R^3\right)=\frac{4\pi}{\ln^2\frac{M}{m}}.$$

Using (2), we find:

$$\frac{1}{Q}a(3)\leq\frac{1}{Q}M\left(E,F;R^3\right)\leq M\left(E^*,F^*;R^3\right)\leq\frac{4\pi}{\ln^2\frac{M}{m}}.$$

As a result, we obtain the estimate

$$\overline{\lim_{r\to 0}}L(\mathrm{r})\leq e^{\left[\frac{4Q\pi}{a(3)}\right]^{\frac{1}{2}}},$$

and f is a quasiconformal mapping.

CONCLUSIONS AND SUGGESTIONS

In this paper, it was proved that the OIM-homeomorphism is quasiconformal without requiring smoothness from the domain of definition to the boundary. and this quasiconformal reflection coefficient was estimated from the given ones. Later, we can consider the issue of reducing the conditions, that is, expanding the class of reflections.

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